

CONSTRUCTIONS OF REGULAR ALGEBRAS  $\mathcal{L}_p^w(G)$ 

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ABSTRACT. Criterion of (Shilov) regularity for weighted algebras  $\mathcal{L}_1^w(G)$  on a locally compact abelian group  $G$  is known by works of Beurling (1949) and Domar (1956). In the present paper this criterion is extended to translation invariant weighted algebras  $\mathcal{L}_p^w(G)$  with  $p > 1$ . Regular algebras  $\mathcal{L}_p^w(G)$  are constructed on any sigma-compact abelian group  $G$ . It was proved earlier by the author that sigma-compactness is necessary (in the abelian case) for the existence of weighted algebras  $\mathcal{L}_p^w(G)$  with  $p > 1$ .

## 1. CRITERION OF REGULARITY

Regular algebras, introduced by G. E. Shilov[13], form an important class of commutative semisimple Banach algebras. Recall their definition. If a commutative Banach algebra  $\mathcal{A}$  is semisimple, the Gel'fand transform allows to identify it with a subalgebra of continuous function on the space of maximal ideals  $\Sigma$ , which is also called the spectrum of  $\mathcal{A}$ . Further,  $\mathcal{A}$  is said to be regular, if the corresponding algebra of functions separates points and closed sets in  $\Sigma$ , i.e. for any closed set  $F \subset \Sigma$  and a point  $x \notin F$  there is  $f \in \mathcal{A}$  such that  $f(x) \neq 0$  and  $f$  vanishes on  $F$ .

Regular algebras possess different interesting properties. For example, if we consider the ideal  $I(F)$  of all functions in  $\mathcal{A}$  that vanish on a closed subset  $F \subset \Sigma$ , the set of common zeros of all functions in  $I(F)$  equals to  $F$  again. If functions with compact support are dense in  $\mathcal{A}$  then every proper closed ideal  $J \subset \mathcal{A}$  is contained in a modular maximal ideal, i.e. has a common zero  $s \in \Sigma$  (it is an abstract form of the Wiener Tauberian theorem, see, e.g., [10, §25D]).

We start with necessary definitions.  $G$  denotes always a locally compact abelian group, all integrals are taken with respect to Haar measure  $\mu$ ,  $p \geq 1$ ,  $1/p + 1/q = 1$  (for  $p = 1$  we put  $q = \infty$ ). A weight is any positive measurable function on  $G$ . The space  $\mathcal{L}_p^w(G)$  with a weight  $w$  is defined as  $\{f : fw \in \mathcal{L}_p(G)\}$ , with the norm  $\|f\|_{p,w} = (\int |fw|^p)^{1/p}$ . Indices  $p, w$  are sometimes omitted. Weights  $w_1, w_2$  are called equivalent

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if for some constants  $C_1, C_2$  locally almost everywhere

$$C_1 \leq \frac{w_1}{w_2} \leq C_2. \quad (1)$$

Equivalent weights define the same weighted space and equivalent norms on it.

Recall sufficient conditions on a weight function for  $\mathcal{L}_p^w(G)$  to be an algebra with the usual convolution:  $(f * g)(s) = \int f(t)g(s-t)dt$ . For  $p = 1$  it is submultiplicativity:

$$w(s+t) \leq w(s)w(t), \quad (2)$$

and for  $p > 1$  the following inequality (locally almost everywhere):

$$w^{-q} * w^{-q} \leq w^{-q}. \quad (3)$$

The space  $\mathcal{L}_p^w(G)$  is translation invariant iff [3] for any  $s \in G$

$$L_s = \operatorname{ess\,sup}_{t \in G} \frac{w(s+t)}{w(t)}. \quad (4)$$

It is obviously sufficient that  $w$  be submultiplicative.

Weighted algebras  $\mathcal{L}_p^w(G)$  are not regular for any weight  $w$ . This is related to the fact that for a quickly growing weight  $w$  Fourier transforms of functions in  $\mathcal{L}_p^w(\mathbb{R})$  form a quasianalytic class, for which uniqueness theorem holds. This result is derived from the Paley-Wiener theorem [11] and is the main tool in the study of Fourier transforms of weighted algebras.

Fix one more notation. For  $x > 0$  denote

$$\log^+ x = \max\{0, \log x\}, \quad \log^- x = \min\{0, \log x\}.$$

A weight  $w$  on the real line is called non-quasianalytic, if

$$\int_{-\infty}^{\infty} \frac{\log^+ w(t)}{1+t^2} dt < \infty, \quad (5)$$

and quasianalytic if this integral diverges.

Beurling [1] proved that (5) is equivalent to regularity of a weighted algebra  $\mathcal{L}_1^w(\mathbb{R})$ . This result was further extended by Domar [2] to the case of any abelian locally compact group, the definition of quasianalyticity slightly changed. According to Domar, a weight  $w$  on a group  $G$  is called non-quasianalytic if for any  $x \in G$

$$\sum_{n=1}^{\infty} \frac{\log^+ w(nx)}{n^2} < \infty. \quad (6)$$

**Theorem 1.1** ([2, th. 2.11]). *Regularity of an algebra  $\mathcal{L}_1^w(G)$  with a weight  $w \geq 1$  is equivalent to the inequality (6).*

The Domar's proof is easily extended to the case  $p > 1$ , if one supposes that the algebra  $\mathcal{L}_p^w(G)$  is translation invariant (theorem 1.3). For non-invariant algebras (6) is not a criterion, see example 1.1. This distinction does not occur in the case  $p = 1$  as algebras  $\mathcal{L}_1^w(G)$  are always translation invariant [9, th. 3.1].

Note that if we multiply the weight of an algebra  $\mathcal{L}_p^w(G)$  by a real-valued character,  $v = \chi w$ , then we get an algebra  $\mathcal{L}_p^v(G)$  which is isometrically isomorphic to the original one. In particular, both algebras are regular or not simultaneously. But the condition (6) may cease to hold after such a transition (consider, e.g., weights  $w(t) = 1 + t^2$  and  $v(t) = e^t(1 + t^2)$  on the real line). It is known [8, th. 3] that it is always possible to find a character  $\chi$  such that the new algebra  $\mathcal{L}_p^v(G)$  with the weight  $v = \chi w$  is contained in  $\mathcal{L}_1(G)$ . The latter inclusion simplifies various details, and particularly, the condition (6) is equivalent to regularity only for algebras satisfying  $\mathcal{L}_p^w(G) \subset \mathcal{L}_1(G)$ . Thus, it should be kept in mind that the inequality (6) is to be tested after previous renorming of the weight.

Proof of theorem 1.3 requires the following lemma.

**Lemma 1.1.** *If  $\mathcal{L}_p^w(G) \subset \mathcal{L}_1(G)$  is an invariant algebra, then*

$$\operatorname{ess\,inf}_G w > 0.$$

*Proof.* By theorem [4, 2.7]  $w$  may be chosen continuous, and by proposition [3, 1.16] the function  $L$  defined in (4) is bounded on every compact set. Let  $D = D^{-1}$  be a compact set of positive measure, and  $N$  such a number that  $L_r \leq N$  for  $r \in D$ . Then for any  $s \in G$ ,  $r \in D$

$$\frac{w(s)}{N} \leq w(s + r) \leq Nw(s). \quad (7)$$

Suppose now that  $\inf_G w = 0$ , i.e.  $w(t_n) = x_n \rightarrow 0$  for some sequence  $t_n \in G$ . Then  $w(t) \leq Nx_n$  for  $t \in t_n + D$ , so that

$$\int_G w^{-q}(t) dt \geq \int_{t_n + D} w^{-q}(t) dt \geq N^{-q} x_n^{-q} \mu(D) \rightarrow +\infty.$$

But by assumption  $\mathcal{L}_p^w(G) \subset \mathcal{L}_1(G)$ , what is by [8, prop. 2] equivalent to the inclusion  $w^{-q} \in \mathcal{L}_1(G)$ . We come to a contradiction.  $\square$

We need also

**Theorem 1.2** ([9, th. 3.3]). *Let  $G$  be an abelian locally compact group, and let  $\mathcal{L}_p^w(G)$  be an invariant algebra. Then  $w$  is equivalent to a submultiplicative function.*

**Corollary 1.1.** *Let  $G$  be an abelian locally compact group, and let  $\mathcal{L}_p^w(G) \subset \mathcal{L}_1(G)$  be an invariant algebra. Then  $w$  is equivalent to a submultiplicative function  $v \geq 1$ .*

*Proof.* Let  $v = Cw$  be equivalent to  $w$  and submultiplicative (th. 1.2). By lemma 1.1  $\delta = \inf_G v > 0$ . If  $\delta \geq 1$ , then  $v$  is the desired function. If  $\delta < 1$ , then it is  $v/\delta$ .  $\square$

**Theorem 1.3.** *Let  $\mathcal{L}_p^w(G) \subset \mathcal{L}_1(G)$  be an invariant algebra. Regularity of this algebra is equivalent to the condition (6).*

*Proof. Sufficiency.* By lemma 1.1 we may assume that  $w \geq 1$  and  $w$  is submultiplicative. If now (6) holds, then  $\mathcal{L}_1^w(G)$  is a regular algebra, and since  $\mathcal{L}_p^w(G)$  is a module over  $\mathcal{L}_1^w(G)$  [9, th. 1.1], this algebra is also regular.

*Necessity.* Suppose the contrary, i.e. that the series (6) diverges for some  $x \in G$ . Let us assume the weight is continuous (corollary 1.1). The closed subgroup  $G_x$  generated by  $x$  is either compact or discrete in  $G$  and therefore isomorphic to  $\mathbb{Z}$  [7, th. 9.1]. In the first case series (6) converges irrespective of regularity. Suppose therefore that  $G_x$  is discrete in  $G$ . Its dual group  $\hat{G}_x = \hat{G}/G_x^\perp$  may be identified with the unit circle, parameter for which we take in  $[-\pi, \pi]$ . Pick a number  $\varepsilon \in (0, \pi)$ . By assumption for any neighborhood of zero  $U \subset \hat{G}$  there exists  $f_0 \in \mathcal{L}_p^w(G)$  such that its support  $\text{supp } \hat{f}_0$  is compact and contained in  $U$ ; we take  $f_0$  such that  $U = (-\varepsilon, \varepsilon) + G_x^\perp$ .

Take now any  $f_1 \in \mathcal{L}_p^w(G)$  with compact support provided that  $\hat{f}_1 \hat{f}_0 \neq 0$ . Convolution  $f = f_0 * f_1$  may be estimated in an arbitrary point  $s$  using submultiplicativity of the weight:

$$\begin{aligned} |f(s)| &\leq \int \left| f_0(t) f_1(s-t) \frac{w(t)w(s-t)}{w(s)} \right| dt \leq \\ &\leq \frac{1}{w(s)} \|f_0 w\|_p \|f_1 w\|_q = \frac{C}{w(s)}, \end{aligned}$$

where the constant  $C$  depends of course on the choice of  $f_0$  and  $f_1$ . Note that  $\text{supp } \hat{f} \subset (-\varepsilon, \varepsilon) + G_x^\perp$ . Restriction of  $f$  onto subgroup  $G_x$ ,  $\varphi_n = f(nx)$ , satisfies

$$|\varphi_n| \leq \frac{C}{w(nx)}, \quad \text{supp } \hat{\varphi} \subset (-\varepsilon, \varepsilon). \quad (8)$$

Indeed, since  $\hat{\varphi} = T_{G_x^\perp} \hat{f}$  [12, 2.7.3] (here  $T$  denotes the averaging operator

$$T_N f(x) = \int_N f_x(t) dt = \int_N f(xt) dt, \quad (9)$$

see [14, §9]), then  $\text{supp } \hat{\varphi} = \text{supp } T_{G_x^\perp} \hat{f} \subset (-\varepsilon, \varepsilon)$ .

Obviously  $\varphi$  can be extended to an entire function of exponential type so that [11, th. X, XII]

$$\int_{-\infty}^{\infty} \frac{\log^- |\varphi(t)|}{1+t^2} dt > -\infty.$$

Submultiplicativity of  $w$  will allow to conclude that the series (6) converges.

Choosing  $\varphi_1 \in \ell_1(\mathbb{Z})$  so that the support  $\text{supp } \hat{\varphi}_1$  is sufficiently small, we may achieve that for  $\tilde{\varphi} = \varphi \cdot \varphi_1$  the condition (8) still holds, moreover,  $\tilde{\varphi} \cdot w \in \ell_1(\mathbb{Z})$ . We may assume therefore that  $\varphi \cdot w \in \ell_1(\mathbb{Z})$ .

We show first that  $\varphi$  as a function on the real line changes little at small translations. Let  $|s| \leq 1$ ,  $n \in \mathbb{Z}$ . Denote  $\xi_s(t) = e^{ist}$ , then

$$\varphi(n+s) = (\xi_s \cdot \hat{\varphi})^\sim(n) = (\hat{\xi}_s * \varphi)(n) \leq \sum_k |\hat{\xi}_s(k) \varphi(n-k)|.$$

Due to (8)

$$|\varphi(n-k)| \leq \frac{C}{w(nx-kx)} \leq \frac{Cw(kx)w(nx-kx)}{w(nx-kx)w(nx)} = \frac{Cw(kx)}{w(nx)}.$$

Convolution with  $\xi_s$  is a multiple of  $\xi_s$ :

$$(\xi_s * \hat{\varphi})(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\varphi}(r) e^{i(t-r)s} dr = \xi_s(t) \varphi(s),$$

i.e.  $\xi_s = \xi_s * \hat{\varphi} / \varphi(s)$   $\hat{\xi}_s = \hat{\xi}_s \cdot \varphi / \varphi(s)$ . Thus,

$$|\varphi(n+s)| \leq \sum_k \left| \hat{\xi}_s(k) \frac{\varphi(k)}{\varphi(s)} \right| \cdot \frac{Cw(kx)}{w(nx)} \leq \frac{C_1}{w(nx)|\varphi(s)|} \|\varphi \cdot w\|_1 \leq \frac{C_2}{w(nx)}$$

for  $s$  in sufficiently small interval  $(-\delta, \delta)$ , as  $\varphi(0) = w(0) \neq 0$  and  $\varphi$  is continuous as a function on the real line.

But now

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\log^+ w(nx)}{n^2} &\leq \frac{1}{2\delta} \sum_{n=1}^{\infty} \int_{n-\delta}^{n+\delta} \frac{\log^+ C_2 - \log^- |\varphi(s)|}{n^2} ds \leq \\ &\leq C_3 - C_4 \int_{-\infty}^{\infty} \frac{\log^- |\varphi(t)|}{1+t^2} dt < +\infty. \end{aligned}$$

□

On the real line regularity of an algebra  $\mathcal{L}_p^w(\mathbb{R})$  is equivalent to convergence of the integral (5) even if the weight is not submultiplicative. A simple proof of sufficiency in the case  $p = 1$  is presented in a paper of E. A. Gorin [6] and may be literally repeated for  $p > 1$ . Necessity may

be proved, for example, in the following way. Let  $f \in \mathcal{L}_p^w(\mathbb{R})$  be such that the support  $\text{supp } \hat{f}$  is compact. Then by Paley-Wiener theorem

$$\int_{-\infty}^{\infty} \frac{\log^- |f(t)|}{1+t^2} dt > -\infty.$$

By definition of the weighted space  $fw = \varphi \in \mathcal{L}_p(\mathbb{R})$ . Since  $\log^+ |\varphi| \leq |\varphi|$ , then  $\log^+ |\varphi| \in \mathcal{L}_p(\mathbb{R})$ . From the other side,  $(1+t^2)^{-1} \in \mathcal{L}_q(\mathbb{R})$  for all  $q \geq 1$ . Thus, from the equality  $w = \varphi/f$  we get:

$$\int_{-\infty}^{\infty} \frac{\log^+ w(t)}{1+t^2} dt \leq \int_{-\infty}^{\infty} \frac{\log^+ |\varphi(t)|}{1+t^2} dt - \int_{-\infty}^{\infty} \frac{\log^- |f(t)|}{1+t^2} dt < +\infty.$$

Submultiplicativity of the weight is essential when we pass from the integral (5) to the series (6), and it would be natural to expect that in its absence condition (6) is no longer equivalent to regularity. It is indeed so what shows the following example.

**Example 1.1.** Consider the unit circle  $\mathbb{T}$  with parameter  $t \in [0, 1)$  and the weight  $w(t) = t^{1/4}$  on it. The space  $\mathcal{L}_2^w(\mathbb{T})$  is an algebra because condition (3) holds. Algebra  $\mathcal{L}_2^w(\mathbb{T})$  is regular because it contains all exponents of type  $f_n(t) = e^{int}$  with Fourier transforms  $\hat{f}_n = \delta_n$ . But the condition (6) does not hold. In order to show it, choose a number  $\alpha \in (0, 1)$  with good rational approximations, e.g.  $\alpha = \sum_{n=1}^{\infty} q_n^{-1}$ , where  $q_1 = 2$ ,  $q_n > 2q_{n-1} \exp(q_{n-1}^2)$ , and all  $q_n$  are integer. Then  $\{q_n \alpha\} < 2q_n/q_{n+1} < e^{-q_n^2}$ . As group operation on  $[0, 1)$  is the fractional part of ordinary sum,

$$\sum_{n=1}^{\infty} \frac{|\log w(n\alpha)|}{n^2} \geq \sum_{n=1}^{\infty} \frac{|\log w(q_n \alpha)|}{q_n^2} = \sum_{n=1}^{\infty} \frac{|\log \{q_n \alpha\}|}{4q_n^2} \geq \sum_{n=1}^{\infty} \frac{|-q_n^2|}{4q_n^2} = +\infty.$$

From results of Domar [2, 1.5] follows

**Theorem 1.4.** *The spectrum of a regular algebra  $\mathcal{L}_p^w(G) \subset \mathcal{L}_1(G)$  coincides with the dual group  $\hat{G}$ .*

**Corollary 1.2.** *Spectrum of a regular algebra  $\mathcal{L}_p^w(G)$  is homeomorphic to the dual group  $\hat{G}$ .*

*Proof.* It follows from the previous theorem and theorem [8, th. 3].  $\square$

Thus, a typical example of a weight of an irregular algebra on the real line is given by  $w(t) = (1+t^2)e^{|t|}$ . Spectrum in this example is the strip  $-1 \leq \Re z \leq 1$  (a proof for  $p = 1$ , valid also for submultiplicative weight and  $p > 1$ , see in [5, §18]). Next example shows that the spectrum of an irregular algebra can be also equal to the dual group.

**Example 1.2.** Let  $w(t) = (1 + t^2) \exp(|t|/\log(e + |t|))$ . Then series (6) diverges for any  $t \neq 0$ , as  $\log w(t) = O(\log t) + |t|/\log(e + |t|)$ . At the same time  $\mathcal{L}_2^w(\mathbb{R})$  is an algebra (inequality (3) is checked straightforwardly). Since  $\lim_{t \rightarrow \pm\infty} t^{-1} \log w(t) = 0$ , spectrum of this algebra is the imaginary axis [5, §18], which may be identified with the dual group  $\mathbb{R}$ .

## 2. CONSTRUCTION OF REGULAR ALGEBRAS

In this section we show that regular weighted algebras with  $p > 1$  may be constructed on every  $\sigma$ -compact abelian group. Since  $\sigma$ -compactness is (in abelian case) necessary for the existence of weighted algebras with  $p > 1$  (th. [9, 1.1]), we see that if weighted algebras do exist on a given group, then there are also regular ones among them. In the case  $p = 1$  there is no problem because the classical algebra  $\mathcal{L}_1(G)$  is regular.

We say that the weight  $w$  grows polynomially if there exists  $d \in \mathbb{N}$  such that for all  $x \in G$

$$w(nx) = O(n^d), \quad n \rightarrow \infty.$$

It is clear that for such a weight Domar's condition (6) holds. On a compactly generated abelian group it makes no trouble to construct such a weight. The task becomes more complicated when the group is not compactly generated, e.g. in the case of rationals  $\mathbb{Q}$  or  $p$ -rationals  $\mathbb{Z}(p^\infty)$ . Separate lemmas 2.2 and 2.1 are devoted to these two groups. The group  $\mathbb{Z}(p^\infty)$  for prime  $p$  is defined as the set  $\{k/p^n : k \in \mathbb{Z}, n \in \mathbb{Z}_+\}$  with addition modulo 1. In theorem 2.1 we show with the help of structure theory that a polynomially growing weight may be constructed on any  $\sigma$ -compact abelian group.

We construct the weight via auxiliary function  $u = w^{-q}$  so that (3) holds. This guarantees that all spaces  $\mathcal{L}_p^w(G)$  are convolution algebras. We list properties that will be required from  $u$  in the following lemmas. This function must be:

- (a) Positive:  $u > 0$
- (b)  $u * u \leq u$
- (c) Even:  $u(x) = u(-x)$
- (d) Decrease polynomially: there exists  $d$  such that  $1/u(nx) = O(n^d)$ ,  $n \rightarrow \infty$  for all  $x$ .

**Lemma 2.1.** *Let  $G$  be equal to the union of its nested finite subgroups  $G_n$ ,  $n \in \mathbb{N}$ :  $G = \cup G_n$ ,  $G_n \subset G_{n+1}$  for all  $n$ . Then on  $G$  exists a weight with properties (a)–(d).*

*Proof.* Let  $|M|$  denote cardinality (which is also Haar measure in this case) of a set  $M \subset G$ . We may construct the weight with the help of any decreasing sequence  $\varphi_n > 0$  provided that  $\varphi = \sum \varphi_n |G_n| < \infty$ . Denote  $U_j = G_j \setminus G_{j-1}$ , assuming  $G_0 = \emptyset$ . Then  $G = \cup U_j$ , and we define  $u$  by  $u|_{U_n} = \varphi_n$ .

Properties (a), (c) are obvious. Since  $1/u(nx)$  is bounded for any  $x$ , the decrease condition (d) is satisfied. Finally, we show that (b) holds. It is clear that

$$(u * u)(x) = \sum_{j=1}^{\infty} \sum_{y \in U_j} \varphi_j u(x - y).$$

Let  $x \in U_n$ ,  $y \in U_j$ . If  $j < n$  then  $x - y \in U_n$ ; if  $j > n$ , then  $x - y \in U_j$ . The set  $U_n$  splits into disjoint union of the sets  $x + U_j$ ,  $j < n$ , and their complement  $U_x$ , for which  $x - U_x \subset U_n$ . Since  $u$  is constant on all these sets, we get that

$$\begin{aligned} (u * u)(x) &= \sum_{j < n} |U_j| \varphi_j \varphi_n + \sum_{j < n} |x + U_j| \varphi_n \varphi_j + |U_x| \varphi_n^2 + \sum_{j > n} |U_j| \varphi_j^2 \leq \\ &\leq 2 \sum_{j=1}^{\infty} |U_j| \varphi_j \varphi_n \leq 2 \varphi_n \varphi = 2u(x) \varphi, \end{aligned}$$

i. e. (b) holds after transition from  $u$  to  $2\varphi u$ .  $\square$

**Corollary 2.1.** *On the group  $\mathbb{Z}(p^\infty)$  for any prime  $p$  there exists a weight with properties (a)–(d).*

**Lemma 2.2.** *On the group  $\mathbb{Q}$  there exists a weight with properties (a)–(d).*

*Proof.* Represent  $\mathbb{Q}$  as a union of nested subgroups  $Q_n = \mathbb{Z}/t_n$ ,  $n \in \mathbb{N}$  — one may take, e.g.,  $t_n = n!$ . Denote  $U_n = Q_n \setminus Q_{n-1}$ , assuming  $Q_0 = \emptyset$ . Let us use functions on  $\mathbb{Z}$

$$\bar{n} = \max\{1, |n|\} \text{ and } \sigma(n) = 1/\bar{n}^2. \quad (10)$$

Let  $C_2$  be such that

$$\sum_{n=-\infty}^{\infty} \frac{1}{\sigma(n)\sigma(m-n)} \leq \frac{C_2}{\sigma(m)} \quad (11)$$

(its existence is easy to verify).

Pick now a decreasing sequence  $\varphi_n > 0$  such that  $\sum \varphi_n t_n = \varphi < \infty$ . For  $q \in U_n$  we put  $u(q) = \varphi_n \sigma(\lfloor q \rfloor)$ , where  $\lfloor x \rfloor$  is the even extension of  $|x|$  from the real half-line.



It is obvious that this weight is even and positive. Any  $q \in \mathbb{Q}$  belongs to some subgroup  $Q_m$ , so that  $u(nq) \geq \varphi_m \sigma(\lfloor nq \rfloor) = \varphi_m / \lfloor nq \rfloor^2$ . Obviously this sequence decreases at most quadratically on  $n$ .

Similarly to the proof of the previous lemma  $(u * u)(q)$  may be split into the sum

$$(u * u)(q) = \sum_{j < n} (S_j + S'_j) + S_n + \sum_{j > n} S_j,$$

where for  $j < n$

$$S_j = \sum_{r \in U_j} \varphi_j \varphi_n \sigma(\lfloor r \rfloor) \sigma(\lfloor q - r \rfloor) = S'_j = \sum_{r \in q + U_j} \varphi_n \varphi_j \sigma(\lfloor r \rfloor) \sigma(\lfloor q - r \rfloor),$$

for  $j > n$

$$S_j = \sum_{r \in U_j} \varphi_j^2 \sigma(\lfloor r \rfloor) \sigma(\lfloor q - r \rfloor)$$

and, finally,

$$S_n = \sum_{r \in U_n \setminus \bigcup_{j < n} (q + U_j)} \varphi_n^2 \sigma(\lfloor r \rfloor) \sigma(\lfloor q - r \rfloor).$$

We estimate now these sums separately. For  $j < n$

$$\frac{S_j}{u(q)} = \varphi_j \varphi_n \sum_{k=0}^{\infty} \sum_{r \in U_j \cap \left( (-k-1, -k] \cup [k, k+1) \right)} \frac{\sigma(\lfloor r \rfloor) \sigma(\lfloor q - r \rfloor)}{\varphi_n \sigma(\lfloor q \rfloor)}.$$

If  $\lfloor r \rfloor = k$ , then  $\lfloor q - r \rfloor$  lies between  $\lfloor q \rfloor - k - 1$  and  $\lfloor q \rfloor - k + 1$ . A rough estimate  $\sigma(l \pm 1) \leq 4\sigma(l)$  holds. Note also that  $|U_j \cap (-k - 1, -k]| = |U_j \cap [k, k + 1)| \leq t_j$ . Moreover,  $\lfloor x \rfloor$  equals 0 on two integer segments. Thus (recall that the notation  $C_2$  is introduced in (11)),

$$\frac{S_j}{u(q)} \leq 8\varphi_j t_j \sum_{k \in \mathbb{Z}} \frac{\sigma(k) \sigma(\lfloor q \rfloor - k)}{\sigma(\lfloor q \rfloor)} \leq 8C_2 \varphi_j t_j \equiv C \varphi_j t_j.$$

Similarly, for  $j \geq n$  we get  $S_j \leq u(q) \cdot C t_j \varphi_j^2 / \varphi_n \leq u(q) \cdot C t_j \varphi_j$  (the latter is true due to decrease of  $\varphi_j$ ). So,

$$(u * u)(q) \leq u(q) \cdot 2 \sum_{j=1}^{\infty} C \varphi_j t_j = u(q) \cdot 2 C \varphi,$$

i. e. (b) holds after transition to  $2 C \varphi u$ .  $\square$

A direct sum  $\oplus_{\alpha \in A} G_{\alpha}$  of groups  $G_{\alpha}$  is usually defined [12, B7] as the subgroup in their direct product  $\prod_{\alpha \in A} G_{\alpha}$  consisting of all elements with finitely many nonzero coordinates. Next lemma constructs

a weight on the countable direct sum by weights on the summands. In fact, the proof remains true even in non-commutative case.

**Lemma 2.3.** *Let  $G_j$ ,  $j \in \mathbb{N}$  be discrete groups with weights  $u_j$  satisfying (a)–(d). On the discrete direct sum  $G = \oplus_{j \in \mathbb{N}} G_j$  there exists a weight  $u$  satisfying the same conditions.*

*Proof.* An element  $x \in G$  is defined by its coordinates  $x_j \in G_j$ . We denote  $s(x) = \{j \in \mathbb{N} : x_j \neq 0\}$  and write also  $s_x$  instead of  $s(x)$ . This set is finite for any  $x$ . Denote also  $G_j^\times = G_j \setminus \{0\}$ . In order to define the weight we need a sequence  $\alpha_j > 0$  and a function  $a : s \rightarrow a_s$  on the set  $\mathcal{F}(\mathbb{N})$  of all finite subsets of the set of natural numbers;  $\alpha_j$  and  $a_s$  will be specified later. Now, define  $u$  as

$$u(x) = a_{s(x)} \prod_{j \in s(x)} \alpha_j u_j(x_j). \quad (12)$$

It is obviously even. We will choose  $a_s$  positive, so that  $u > 0$ . Note also that  $s(nx) = s(x)$  for all  $x \in G$ , and  $j$ -th coordinate of  $nx$  is  $nx_j$ . Thus, decrease condition (c) for  $u$  follows trivially from the same condition holding for every  $u_j$ .

For (b), the following inequalities are sufficient. Numbers  $\alpha_j$  should be small:

$$\forall j \in \mathbb{N} \quad 0 < \alpha_j < 1, \quad (13)$$

$$\prod_{j=1}^{\infty} (1 + \alpha_j^2 \cdot (u_j * u_j)(0)) < 2, \quad (14)$$

$$\prod_{j=1}^{\infty} (1 + \alpha_j) < 2. \quad (15)$$

It is obvious that such  $\alpha_j$  exist. From  $a$  we require that for all  $s \in \mathcal{F}(\mathbb{N})$

$$0 < a_s \leq 1, \quad (16)$$

$$a_{s \cup v} \leq a_s \quad \forall v \in \mathcal{F}(\mathbb{N}), \quad (17)$$

$$\sum_{v \subset s} \frac{a_v a_{s \setminus v}}{a_s} \leq \frac{1}{4}. \quad (18)$$

One may take  $a_s = \varepsilon_1 / (\sum_{j \in s} j!)$  with  $\varepsilon_1 = e^{-2}/8$ . For the empty set we put  $a_\emptyset = \varepsilon_1$ . Properties (16), (17) hold obviously. Check now (18). Note that for nonempty  $v \subsetneq s$  we have  $a_s^{-1} = a_v^{-1} + a_{s \setminus v}^{-1}$ . Therefore

$$\sum_{v \subset s} \frac{a_v a_{s \setminus v}}{a_s} = 2a_\emptyset + \sum_{v \subsetneq s} a_v a_{s \setminus v} \left( \frac{1}{a_v} + \frac{1}{a_{s \setminus v}} \right) = 2 \sum_{v \subsetneq s} a_v.$$

Further,  $a_v = \varepsilon_1 / (\sum_{j \in v} j!) \leq \varepsilon_1 / (\sup v)!$ . assuming  $\sup \emptyset = 0$ . Thus,

$$\sum_{\substack{v \subseteq s \\ v \neq s}} a_v \leq \sum_{\substack{v \subseteq s \\ v \neq s}} \frac{\varepsilon_1}{(\sup v)!} \leq \sum_{k=0}^{\sup s} \frac{\varepsilon_1 2^k}{k!} \leq \varepsilon_1 e^2 = \frac{1}{8},$$

whence (18) follows.

Estimate now the convolution  $u * u$ . Take  $x \in G$ . Every  $x' \in G$  may be represented then in the form  $x' = y + z$ , where  $s_y \subset s_x$ ,  $s_z \subset \mathbb{N} \setminus s_x$ . According to the definition (12),

$$u(x') = u(y + z) = u(y) \frac{a_{s_y \cup s_z}}{a_{s_y}} \prod_{j \in s_z} \alpha_j u_j(z_j) \stackrel{(17)}{\leq} u(y) \prod_{j \in s_z} \alpha_j u_j(z_j).$$

As  $s_{x-y-z} = s_{x-y} \cup s_z$  and  $(x - y - z)_j = -z_j$  for  $j \in s_z$ , using the fact that all weights are even,

$$\begin{aligned} u(x - x') &= u(x - y - z) = u(x - y) \frac{a_{s_{x-y} \cup s_z}}{a_{s_{x-y}}} \prod_{j \in s_z} \alpha_j u_j(z_j) \leq \\ &\stackrel{(17)}{\leq} u(x - y) \prod_{j \in s_z} \alpha_j u_j(z_j). \end{aligned}$$

Consider subgroup  $Q_x = \prod_{j \in s_x} G_j$  extracted by the support of  $x$ . An element  $y$  belongs to this subgroup iff  $s_y \subset s_x$ . Using the above inequalities, we get:

$$\begin{aligned} u * u(x) &= \sum_{x' \in G} u(x') u(x - x') \leq \sum_{y \in Q_x} u(y) u(x - y) \sum_{\substack{z \in G \\ s_z \subset \mathbb{N} \setminus s_x}} \prod_{j \in s_z} \alpha_j^2 u_j^2(z_j) \leq \\ &\leq \sum_{y \in Q_x} u(y) u(x - y) \prod_{j=1}^{\infty} (1 + \alpha_j^2 \cdot (u_j * u_j)(0)) \stackrel{(14)}{<} 2 \sum_{y \in Q_x} u(y) u(x - y). \end{aligned}$$

It remains thus to estimate sum over  $Q_x$ . We will write further  $s$  instead of  $s_x$ . Let  $y \in Q_x$  and  $s_y = v$ . The difference  $x - y$  has some support  $s_{x-y}$ . Denote  $r = v \cap s_{x-y}$ . Then coordinates off  $r$  are determined uniquely:

$$\begin{cases} y_j = x_j, & (x - y)_j = 0 & \text{for } j \in v \setminus r, \\ y_j = 0, & (x - y)_j = -x_j & \text{for } j \in s \setminus v. \end{cases}$$

Taking into account all weights are even, we note that in the following fraction after cancellation it remains only

$$\begin{aligned} \frac{u(y)u(x-y)}{u(x)} &= \frac{a_v a_{r \cup (s \setminus v)}}{a_s} \prod_{j \in r} \alpha_j \frac{u_j(y_j)u_j(x_j - y_j)}{u_j(x_j)} \leq \\ &\stackrel{(17)}{\leq} \frac{a_v a_{s \setminus v}}{a_s} \prod_{j \in r} \alpha_j \frac{u_j(y_j)u_j(x_j - y_j)}{u_j(x_j)}. \end{aligned}$$

Estimate sum over  $Q_x$ :

$$\begin{aligned} \sum_{y \in Q_x} \frac{u(y)u(x-y)}{u(x)} &= \sum_{v \subset s} \sum_{s(y)=v} \frac{u(y)u(x-y)}{u(x)} = \\ &= \sum_{v \subset s} \frac{a_v a_{s \setminus v}}{a_s} \sum_{r \subset v} \sum_{\substack{s(y)=r \\ y_j \neq x_j}} \prod_{j \in r} \alpha_j \frac{u_j(y_j)u_j(x_j - y_j)}{u_j(x_j)} = \\ &= \sum_{v \subset s} \frac{a_v a_{s \setminus v}}{a_s} \sum_{r \subset v} \prod_{j \in r} \alpha_j \sum_{0 \neq y_j \neq x_j} \frac{u_j(y_j)u_j(x_j - y_j)}{u_j(x_j)} \stackrel{(b)}{\leq} \sum_{v \subset s} \frac{a_v a_{s \setminus v}}{a_s} \sum_{r \subset v} \prod_{j \in r} \alpha_j = \\ &= \sum_{v \subset s} \frac{a_v a_{s \setminus v}}{a_s} \prod_{j \in v} (1 + \alpha_j) \stackrel{(15)}{\leq} 2 \sum_{v \subset s} \frac{a_v a_{s \setminus v}}{a_s} \stackrel{(18)}{\leq} \frac{1}{2}. \end{aligned}$$

Thus, for all  $x \in G$

$$u * u(x) \leq u(x),$$

i.e. (b) holds.  $\square$

We can now construct a weight on any countable group, using the structure theorem.

**Lemma 2.4.** *On every countable discrete abelian group  $G$  exists a weight with properties (a)–(d).*

*Proof.* It is known that  $G$  may be embedded as a subgroup into a divisible group  $H$ , which is also countable. By structure theorem for abelian groups [7, suppl. A]  $H$  is isomorphic to the direct sum of copies of rationals  $\mathbb{Q}$  and  $p$ -rational numbers  $\mathbb{Z}(p^\infty)$  with prime  $p$ . Since  $H$  is countable, the number of summands is countable. By lemmas 2.2, 2.1 on all summands exist weights with properties (a)–(d). By lemma 2.3 the same is true for  $H$ . If we restrict now the weight from  $H$  onto  $G \subset H$ , properties (a)–(d) with remain true.  $\square$

**Theorem 2.1.** *On every  $\sigma$ -compact locally compact abelian group  $G$  for every  $p > 1$  there exists an invariant algebra  $\mathcal{L}_p^w(G)$  with polynomially growing weight.*

*Proof.* By structure theorem ([7, 24.30])  $G$  is topologically isomorphic to  $\mathbb{R}^d \times H$ , where  $d$  is a nonnegative integer and  $H$  contains a compact open subgroup  $E$ . The quotient group  $H/E$  is discrete, and by  $\sigma$ -compactness of  $G$  it is countable.

Let functions  $u_R$  on  $\mathbb{R}^d$  and  $u_H$  on  $H/E$  satisfy (a)–(d). Existence of  $u_H$  is proved in lemma 2.4, and  $u_R$  may be taken equal to

$$u_R(x) = \frac{1}{(1 + x_1^2) \cdots (1 + x_d^2)}.$$

The function  $u$  on  $G$ ,

$$u(r, h) = u_R(r)u_H(h + E),$$

also satisfies (a)–(d). But now for any  $p > 1$  the space  $\mathcal{L}_p^w(G)$  with the weight  $w = u^{-q}$  is an algebra, its weight growing polynomially. Moreover,  $u_H^{-q}$  is submultiplicative because it defines an algebra [9, remark 3.1]. We see that  $u^{-q}$  and  $w$  are also submultiplicative, thus the algebra  $\mathcal{L}_p^w(G)$  is invariant.  $\square$

Now it follows

**Theorem 2.2.** *On every  $\sigma$ -compact locally compact abelian group for any  $p > 1$  there exists a regular algebra  $\mathcal{L}_p^w(G)$ .*

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